ON THE METHOD OF CHARACTERISTICS IN THE THEORY OF MOTION OF A MULTICOMPONENT MEDIUM

(O METODE KHARAKTERISTIK V TEORII DVIZHENIIA MNOGOKOMPONENTNOI SREDY)

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The equations of motion and continuity for the multicomponent medium in a cartesian coordinate system have been obtained in [1]. Analogous equations may be written in curvilinear coordinates.

We will confine ourselves to the consideration of plane, cylindrical and spherical waves.* The number of the component of the mixture is equal to N.

Using the cartesian, cylindrical and spherical coordinate systems, for the consideration of the three cases, as usual we have in general the equations of motion and continuity (neglecting the influence of the mass forces):

$$\frac{\partial v_n}{\partial t} + v_n \frac{\partial v_n}{\partial x} + \frac{1}{\rho_n^{\circ}} \frac{\partial p}{\partial x} - \frac{1}{\rho_n} \sum_{j=1}^{N} K_{jn} (v_j - v_n) = 0 \qquad (1)$$

$$\frac{\partial \rho_n}{\partial t} + v_n \frac{\partial \rho_n}{\partial x} + \rho_n \frac{\partial v_n}{\partial x} + \frac{s}{x} \rho_n v_n = 0 \quad (n = 1, \ldots, N) \qquad (2)$$

where p is the pressure, v_n, ρ_n^0, ρ_n are the velocity, the actual and the partial** density respectively of the n-th component, and K_{jn} is the function of the interaction of the n-th component with j-th component.

* The case of plane waves as applied to the two-component medium is considered in [1].

** Partial density of the n-th component in a given volume of the mixture is called the density which the n-th component would have if it alone occupied all of that volume.

For plane waves $s = 0$, for the cylindrical and spherical waves $s = 1$ and $s = 2$ correspondingly. In the cases of axial and spherical symmetry we denote the magnitude of the radius-vector in the corresponding coordinate system by x.

To system (1), (2) we add the equation

$$\sum_{n=1}^{N} \frac{\rho_n}{\overset{\circ}{\rho}_n} = 1 \tag{3}$$

which results from the fact that the magnitude ρ_n/ρ_n^0 is the part of the unit volume of the medium occupied by the n-th component. We also add the relationships

$$p = f_n(\overset{\circ}{\rho}_n, p_0, \overset{\circ}{\rho}_{0n}) \qquad (n = 1, \ldots, N) \tag{4}$$

where p_0, ρ_{0n}^0 are initial values of the pressure and the actual density of the n-th component.

In this way, we have a closed system of $3N + 1$ equations.

We will derive the equations of characteristics of this system. For the parameters specified along the characteristics, we may write

$$dv_n = \frac{\partial v_n}{\partial t} dt + \frac{\partial v_n}{\partial x} dx, \qquad d\rho_n = \frac{\partial \rho_n}{\partial t} dt + \frac{\partial \rho_n}{\partial x} dx \qquad (n = 1, \ldots, N)$$

Hence it follows

$$\frac{\partial v_n}{\partial t} = \frac{dv_n}{dt} - \frac{\partial v_n}{\partial x}\frac{dx}{dt}, \qquad \frac{\partial \rho_n}{\partial t} = \frac{d\rho_n}{dt} - \frac{\partial \rho_n}{\partial x}\frac{dx}{dt} \qquad (n = 1, \ldots, N)$$

Substituting these expressions into (1) and (2), we have

$$\frac{dv_n}{dt} + \frac{\partial v_n}{\partial x}(v_n - x_t) + \frac{1}{\overset{\circ}{\rho}_n}\frac{\partial p}{\partial x} - \frac{1}{\rho_n}\sum_{j=1}^{N} K_{jn}(v_j - v_n) = 0 \qquad (n = 1, \ldots, N)$$

$$\frac{d\rho_n}{dt} + \frac{\partial \rho_n}{\partial x}(v_n - x_t) + \rho_n \frac{\partial v_n}{\partial x} + \frac{s}{x}\rho_n v_n = 0$$

where

$$x_t = \frac{dx}{dt}$$

Eliminating $\partial v_n/\partial x$ from these expressions, we obtain

$$\frac{\partial \rho_n}{\partial x} = \frac{1}{(v_n - x_t)^2}\left[\frac{\rho_n}{\overset{\circ}{\rho}_n}\frac{\partial p}{\partial x} + \rho_n \frac{dv_n}{dt} - (v_n - x_t)\left(\frac{d\rho_n}{dt} + \frac{s}{x}\rho_n v_n\right) - \sum_{j=1}^{N} K_{jn}(v_j - v_n)\right] \tag{5}$$

Taking the partial* derivative of (3) with respect to x and considering the relationship

$$\frac{\partial \rho_n{}^\circ}{\partial x} = \frac{1}{a_n{}^2}\frac{\partial p}{\partial x}$$

where $a_n{}^2 = dp/d\rho_n{}^0$ is a known function if relationship (4) is given, we have

$$\sum_{n=1}^{N}\left(\frac{1}{\rho_n{}^\circ}\frac{\partial \rho_n}{\partial x} - \frac{\rho_n}{\rho_n{}^{\circ 2} a_n{}^2}\frac{\partial p}{\partial x}\right) = 0$$

Substituting (5) into this equation, we obtain

$$\frac{\partial p}{\partial x}\sum_{n=1}^{N}\frac{\rho_n}{\rho_n{}^{\circ 2}}\left[\frac{1}{(v_n - x_t)^2} - \frac{1}{a_n^2}\right] =$$

$$= -\sum_{n=1}^{N}\frac{1}{\rho_n{}^\circ (v_n - x_t)^2}\left[\rho_n\frac{dv_n}{dt} - (v_n - x_t)\left(\frac{d\rho_n}{dt} + \frac{s}{x}\rho_n v_n\right) - \sum_{j=1}^{N} K_{jn}(v_j - v_n)\right]$$

Equating the coefficient of $\partial p/\partial x$ and the right-hand side of the last equation, we obtain the equations of the characteristics:

$$\sum_{n=1}^{N}\frac{\rho_n}{\rho_n{}^{\circ 2}}\left[\frac{1}{(v_n - x_t)^2} - \frac{1}{a_n{}^2}\right] = 0 \tag{6}$$

$$\sum_{n=1}^{N}\frac{1}{\rho_n{}^\circ (v_n - x_t)^2}\left[\rho_n\frac{dv_n}{dt} - (v_n - x_t)\left(\frac{d\rho_n}{dt} + \frac{s}{x}\rho_n v_n\right) - \sum_{j=1}^{N} K_{jn}(v_j - v_n)\right] = 0 \tag{7}$$

It may be shown that equation (6) coincides with the equation which governs the velocity of wave propagation of a weak discontinuity in the mixture. Thus, as in the case of a one-component medium, the characteristic surfaces of the system of equations (1)-(4) and the surfaces of propagation of weak discontinuities are identical.

Investigation of equation (6) shows that the quantity of real characteristics for a given number of components of the mixture may vary (from 2 to $2N$) according to the parameters of the medium. Also some characteristics may prove to be imaginary, or there are coincident multiple real

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* Note that equation (3) may be either partially differentiated with respect to t (at a given fixed point in space), or partially with respect to the coordinates (at a given moment of time). Taking the total derivatives with respect to t does not have a physical meaning in the general case, as the particles of different components which at a given moment were at one point in space may be at different points at the next moment.

characteristics. Thus it is not always possible to apply the usual method of characteristics to the solution of problems of non-static motions of mixtures, in contrast to the one-component medium. For the numerical solution of the problems it is necessary either to supplement the equations of characteristics with initial equations, written as finite differences, or to use only the connective equations of the initial system.

Let us note that when the mass forces F $(n = 1, \ldots, N)$ are taken into consideration in the case of the one-component motion of the medium along the direction of action of these forces, an additive F appears on the left-hand side of equation (1), and equation (7) then assumes the form

$$\sum_{n=1}^{N} \frac{1}{\rho_n{}^{\circ}(v_n - x_t)^2}\left[\rho_n \frac{dv_n}{dt} - (v_n - x_t)\frac{d\rho_n}{dt} - \sum_{j=1}^{N} K_{jn}(v_j - v_n) - \rho_n F_n\right] = 0$$

Consider the case of a two-component medium. The equations of characteristics are then written in the form

$$\frac{\rho_1}{\rho_1{}^{\circ 2}}(v_2 - x_t)^2 + \frac{\rho_2}{\rho_2{}^{\circ 2}}(v_1 - x_t)^2 - \left(\frac{\rho_1}{\rho_1{}^{\circ 2}a_1{}^2} + \frac{\rho_2}{\rho_2{}^{\circ 2}a_2{}^2}\right)(v_1 - x_t)^2(v_2 - x_t)^2 = 0 \tag{8}$$

$$\frac{\rho_2\rho_1{}^{\circ}(v_1 - x_t)^2}{\rho_1\rho_2{}^{\circ}(v_2 - x_t)^2} = -\frac{b_1}{b_2} \tag{9}$$

where

$$b_i = (v_i - x_t)\left(\frac{1}{\rho_i}\frac{d\rho_i}{dt} + \frac{sv_i}{x}\right) - \frac{dv_i}{dt} \pm \frac{K(v_2 - v_1)}{\rho_i} \qquad (i = 1, 2)$$

where the plus sign corresponds to $i = 1$ and the minus sign to $i = 2$.

Introducing the notations

$$z = x_t - v_1, \qquad y = x_t - v_2 \tag{10}$$

equation (8) will be represented in the form

$$z = \pm\sqrt{\frac{Aa_1{}^2y^2}{(A + a_1{}^2/a_2{}^2)\,y^2 - a_1{}^2}} \qquad \left(A = \frac{\rho_1\rho_2{}^{\circ 2}}{\rho_2\rho_1{}^{\circ 2}}\right) \tag{11}$$

From (10) we have

$$z = y + \Delta v \qquad (\Delta v = v_2 - v_1) \tag{12}$$

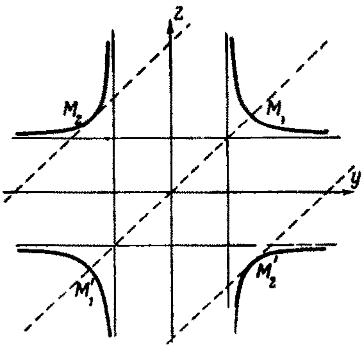
The graphs of functions (11) and (12) are represented in the figure.

The points of intersection of the curves (11) with the straight lines (12) yield the values y and z, from which the x quantities are determined which correspond to the real characteristics. We see that with the increase of the quantity $|\Delta v|$ from zero to some critical value $|\Delta v_k|$, corresponding to the points M_2 and $M_2{}'$, there are two different real characteristics, while for $|\Delta v| > |\Delta v_k|$ there are four different real characteristics. In the case $|\Delta v| = |\Delta v_k|$, two real characteristics (out of four) coincide.

Thus, the application of the usual method of characteristics in the case of two-component medium is possible for the values $|\Delta v| > |\Delta v_k|$.

We will show, however, that $|\Delta v_k|$ is by no means of insignificant magnitude : it is of the order of the velocity of propagation of the perturbations in a two-component medium, and consequently, in many cases $|\Delta v| < |\Delta v_k|$.

Indeed, when calculating the derivative dz/dy from (11) and considering that at the points M_2 and M_2' $dz/dy = 1$, because of (12), for the given points we obtain



$$y_k = (x_t - v_2)_k = \pm a_1 \sqrt{\frac{A^{1/3}+1}{A + a_1^2/a_2^2}}$$

$$z_k = (x_t - v_1)_k = \mp a_1 A^{1/3} \sqrt{\frac{A^{1/3}+1}{A + a_1^2/a_2^2}}$$

These expressions lead to

$$\Delta v_k = \pm B a_*$$

where

$$B = \sqrt{\frac{(A^{1/3}+1)^3}{A+1}}, \quad a_* = a_1 \sqrt{\frac{A+1}{A + a_1^2/a_2^2}}$$

The quantity a_* is the wave propagation velocity of weak discontinuities in an initially stagnant two-component medium (in the case $v_1 = v_2 = v_0$ from (8) we obtain $x_1 - v = \pm a_*$). It is easy to verify that the coefficient B varies within the limits

$$1 \leqslant B \leqslant 2$$

if

$$0 \leqslant \rho_1/\rho_2 \leqslant 1$$
$$\infty \geqslant \rho_1/\rho_2 \geqslant 1$$

Consequently, for the evaluation of the values of $|\Delta v_k|$ we have

$$a_* < |\Delta v_k| < 2a_*$$

Thus, in the case of a two-component medium, in may problems of current practical interest, the number of real characteristics is equal to two, i.e. the usual method of characteristics cannot be applied.

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